COMPACT OPERATORS

Recall (X, d) metric space

- $A \subseteq X$ is PRECOMPACT if $\forall \varepsilon > 0$ $\exists$ finite ε-net, i.e. finitely many balls $B_\varepsilon(x_1), \dots, B_\varepsilon(x_N)$ of radius ε covering A: $A \subseteq \bigcup_{i=1}^{N} B_\varepsilon(x_i)$

$B^X_r(x) = \{ y \in X \mid \|y - x\|_X \leq r \}$

- $A \subset X$ is RELATIVELY COMPACT if $\overline{A}$ is compact
- (X, d) metric, complete

precompact $\iff$ relatively compact

Def (compact op) X, Y Banach, $K : X \to Y$ is COMPACT if the image of the unit ball is precompact, i.e.

$$\overline{K B^X_1(0)} \text{ is compact}$$

$$\mathcal{K}(X, Y) = \{ \text{compact op } X \to Y \}$$

ELEMENTARY EXAMPLES

- zero op
- (IMPORTANT EXAMPLE) bounded op with finite dim range

Indeed let $T \in \mathcal{L}(X, Y)$ $\dim(\operatorname{Ran} T) < +\infty$

$T B^X_1(0)$ is bd set : $\|Tx\| \leq \|T\| \, \|x\| \leq \|T\|$

$\rightsquigarrow$ $\overline{T B^X_1(0)}$ closed bd set in a finite dim space $\rightsquigarrow$ compact

⓪ $\mathbb{I}: X \to X$ is not compact if $\dim X = +\infty$

<u>Rem</u> (1) If $\overline{K B_1^X(0)}$ is compact $\Rightarrow$ $\overline{K\, B_R^X(0)}$ compact $\forall R>0$

(by linearity of K)

(2) T compact IFF $\{x_n\}$ bd $\Rightarrow$ $\{Tx_n\}$ convergent sub

MAIN PROPERTIES OF COMPACT OPERATORS

<u>Prop</u> X, Y, Z Banach

(1) $\mathcal{K}(X,Y)$ closed (in the norm topology) lin. sub in $\mathcal{L}(X,Y)$

(2) $\left.\begin{array}{l} A \in \mathcal{K}(X,Y), \quad B \in \mathcal{L}(Y,Z) \\ \text{or} \\ A \in \mathcal{L}(X,Y), \quad B \in \mathcal{K}(Y,Z) \end{array}\right\} \Rightarrow BA \in \mathcal{K}(X,Z)$

(3) K compact $\iff$ K' compact $(K' \in \mathcal{L}(Y^*, X^*))$

(4) K compact, then $x_n \rightharpoonup x \Rightarrow Kx_n \to Kx$

(it maps weakly conv. seq in norm conv. seq)

<u>proof</u> (1) $\mathcal{K}(X,Y)$ is linear sub (exercise!)

Take $(K_n)_{n\geq 1} \subseteq \mathcal{K}(X,Y)$, $K \in \mathcal{L}(X,Y)$ s.t.

$\| K_n - K \| \xrightarrow{n\to\infty} 0$. We want to prove K compact

We show that $K B_1^X(0)$ is precompact, in particular

$\forall \varepsilon > 0$ we construct an ε-net.

Take $\varepsilon > 0$, then $\exists n : \| K_n - K \| \leq \frac{\varepsilon}{2}$

But K_n is compact $\rightsquigarrow$ $K_n B_1^X(0)$ has $\frac{\varepsilon}{2}$-net :

$$K_n B_1^X(0) \subseteq B_{\frac{\varepsilon}{2}}(y_1) \cup \dots \cup B_{\frac{\varepsilon}{2}}(y_d)$$

where $y_1, \dots, y_d \in Y$

Take Kx with $x \in B_1^X(0)$, we want to find y_j with $\| Kx - y_j \| \leq \varepsilon$

Fix $x \in B_1^X(0)$

$$\| Kx - y_j \| \leq \| Kx - K_n x \| + \| K_n x - y_j \|$$

choose y_j so that $K_n x \in B_{\frac{\varepsilon}{2}}(y_j)$

$$\leq \| K - K_n \| \, \| x \| + \frac{\varepsilon}{2}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$Kx \in \bigcup_{i=1}^{d} B_{\frac{\varepsilon}{2}}(y_i) \qquad \forall x \in B_1^X(0)$$

$\rightsquigarrow$ $K B_1^X(0)$ is precompact

(2) $(x_n)_{n \geq 1} \subset X$ bd. A compact $\Rightarrow$ $(Ax_n)_{n \geq 1}$ has conv. sub

$\Rightarrow$ $(BAx_n)_{n \geq 1}$ has conv. sub (B bd operator)

(3) $\Rightarrow$) Assume $K : X \to Y$ compact, want to prove $K' : Y^* \to X^*$ compact

Take $(f_n)_{n \geq 1} \subseteq B_1^{Y^*}(0)$, want $(K^* f_n)_{n \geq 1}$ has conv. sub in X^*.

In particular, want to prove $\exists$ subseq $(K^* f_{n_k})_{n \geq 1}$ which is Cauchy in X^*

$$\| K^* f_{n_k} - K^* f_{n_\ell} \|_{X^*} = \sup_{\|x\| \le 1} | (K^* f_{n_k})(x) - (K^* f_{n_\ell})(x) |$$

$$= \sup_{\|x\| \le 1} | f_{n_k}(Kx) - f_{n_\ell}(Kx) |$$

We want to prove that $f_{n_k} \circ K : \overline{B_1^X(0)} \to \mathbb{C}$ has a conv. sub.

$\leadsto$ we show $\varphi_n : \overline{K B_1^X(0)} \to \mathbb{C}$, $x \to f_n(x)$, i.e. $\varphi_n = f_n|_{\overline{K B_1^X(0)}}$

has a conv. sub

Define $M = \{ \varphi_n : \overline{K B_1^X(0)} \to \mathbb{C}, \ n = 1, 2, 3, \dots \}$

each φ_n is continuous

$$M \subseteq C(\underbrace{\overline{K B_1^X(0)}}_{\text{metric compact set}})$$

<u>Thm</u> (Ascoli–Arzelà) D compact metric space
A set $\mathcal{F} \subseteq C(D)$ has compact closure in $C(D)$ if
(i) $\mathcal{F}$ is bd
(ii) $\mathcal{F}$ is uniformly equicontinuous:
$\forall \varepsilon > 0, \exists \delta > 0$: $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{F}$

Show that M fulfills A-A assumptions: $D := \overline{K B_1^X(0)}$

<u>M bd</u>

$$\sup_{x \in D} |\varphi_n(x)| = \sup_{x \in D} |f_n(x)| \le \overbrace{\|f_n\|_{Y^*}}^{\le 1} \overbrace{\sup_{x \in D} |x|}^{\le C} \le C \quad \forall n \checkmark$$

<u>equicontinuity</u>

$$|\varphi_n(x_1) - \varphi_n(x_2)| = |f_n(x_1) - f_n(x_2)| = |f_n(x_1 - x_2)| \le \|f_n\|_{Y^*} \|x_1 - x_2\| \le \|x_1 - x_2\|$$

By A-A we find a subseq $\varphi_{n_k} \to \varphi$ in $C^0(D)$

$\rightsquigarrow$ (φ_{n_k}) Cauchy in $C^0(D)$ $\rightsquigarrow$ $(f_{n_k} \circ K)$ Cauchy

in $C(B_1^{X^*}(0), \|\ \|_\infty)$

$\Leftarrow$) $\underline{K^* \text{ compact}}$. By the previous point $K^{**} \in K(X^{**}, Y^{**})$

$K^{**}(B_1^{X^{**}}(0))$ is compact in Y^{**}, so is

$\overline{K^{**} B_1^X(0)}$ (since $B_1^X(0) \hookrightarrow B_1^{X^{**}}(0)$)

Denote $J_X : X \to X^{**}$, $J_Y : Y \to Y^{**}$ (isometries)

<u>claim</u> $K^{**}(J_X x) = J_Y(Kx)$ in Y^{**}

$\Rightarrow$ $K^{**}(J_X B_1^X(0)) = J_Y K(B_1^X(0))$

$\rightsquigarrow$ $J_Y K(B_1^X(0))$ is precompact in Y^{**}

J_Y isometry $\rightsquigarrow$ (exercise) $K B_1^X(0)$ is precompact in Y

<u>proof of claim</u> $\ell \in Y^*$ $K^{**}(J_X x)[\ell] = (J_X x)[K^* \ell]$

$= (K^* \ell)(x) = \ell(Kx)$

$= (J_Y Kx)(\ell)$

(4) $x_n \rightharpoonup x$ $\Rightarrow$ $Kx_n \to Kx$

<u>step 1</u> $Kx_n \rightharpoonup Kx$: indeed $\ell \in Y^*$. $K^* \ell \in X^*$

$\ell(Kx_n) - \ell(Kx) = (K^* \ell)x_n - (K^* \ell)x \longrightarrow 0$

<u>step 2</u> $Kx_n \to Kx$. By contradiction is false, $\exists\, \varepsilon > 0$ and

$\{Kx_{n_k}\}$ with $\|Kx_{n_k} - Kx\| \geq \varepsilon$

but $x_n \to x \rightsquigarrow \|x_n\| \le C \quad \forall n$

$\rightsquigarrow (Kx_{n_k})$ has conv. sub $\rightsquigarrow Kx_{n_k} \to y \in Y$

for some $y \neq Kx$

but $Kx_{n_k} \to y$ ↯

$\square$

Cor 1 Let $\dim X = +\infty$ and $T \in \mathcal{K}(X)$

then T is not invertible

proof B.C. $\exists\, T^{-1} \in \mathcal{L}(X)$, then $\mathbb{1}_X = \underbrace{T}_{\in \mathcal{K}} \circ \underbrace{T^{-1}}_{\in \mathcal{L}}$

is compact ↯

Cor 2 $(T_n)_{n\geq 1}$ seq of op with finite range and bounded and $T \in \mathcal{L}(X,Y)$ s.t. $\|T_n - T\|_{\mathcal{L}(X,Y)} \to 0$, $n \to +\infty$

Then T is compact

proof T_n are compact and $\mathcal{K}(X,Y)$ closed

Q: What about the converse statement?

Given T compact, $\exists\, (T_n)_{n\geq 1} \in \mathcal{L}(X,Y)$ finite range op with $\|T - T_n\| \xrightarrow{n\to+\infty} 0$?

In general No, but true in Hilbert space

Prop H Hilbert, separable, T compact. Then $\exists\, (T_n)_{n\geq 1}$ bounded, finite range s.t. $\|T_n - T\| \to 0$, $n \to \infty$

proof T compact $\iff T B_1^H(0)$ precompact

$\forall \varepsilon > 0 \;\exists\; \varepsilon$-net : $T B_1^H(0) \subseteq \bigcup_{j=1}^{d} B_\varepsilon(y_j)$

with $y_1, \dots, y_d \in Y$

Let $G = \text{span}(y_1, \dots, y_d)$, then G closed, $\dim G < +\infty$
then
$P_G : H \to G$ the orthog. proj on G

Define $T_\varepsilon := P_G \circ T$. Clearly $T_\varepsilon \in \mathcal{L}(H)$
and finite rank
We need to check $\| T_\varepsilon - T \| \leq 2\varepsilon$

Indeed take arbitrary $x \in B_1^H(0)$, then $\exists\, y_j \in G$ st
$\| Tx - y_j \| \leq \varepsilon$

$$\Rightarrow \| T_\varepsilon x - Tx \| \leq \| T_\varepsilon x - y_j \| + \| y_j - Tx \|$$

$$\leq \| P_G \circ Tx - y_j \| + \varepsilon$$

$y_j \in G$
$y_j = P_G y_j$ $\leq \| P_G (Tx - y_j) \| + \varepsilon$

$$\leq \| Tx - y_j \| + \varepsilon \leq 2\varepsilon \quad \forall x$$

$$\Rightarrow \sup_{\|x\| \leq 1} \| (T_\varepsilon - T) x \| \leq 2\varepsilon$$ $\boxtimes$

Q: Is any finite range op bounded?

$T : \ell^2 \to \ell^2$
$\vec{e}_n \to (n \vec{e}_1)_{n \geq 1}$

EXAMPLES

(i) Diagonal operators on seq space:

$$T: \ell^2 \longrightarrow \ell^2$$
$$(x_1, x_2, x_3, \dots) \longmapsto (\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \dots)$$

where $(\alpha_n)_{n\geq 1} \subseteq \mathbb{C}$. If $\sup_n |\alpha_n| < +\infty$

$\Rightarrow T$ is $\mathcal{L}(\ell^2)$.

If $\alpha_n \longrightarrow 0$ when $n \to \infty$ $\Rightarrow T$ is compact.

proof:
$$T_N : \ell^2 \longrightarrow \ell^2$$
$$(x_1, x_2, \dots) \longmapsto (\alpha_1 x_1, \dots, \alpha_N x_N, 0, 0, \dots)$$

$T_N \in \mathcal{L}(\ell^2)$ and finite range (T_N compact)

$$\| T - T_N \|^2 = \sup_{\|x\| \leq 1} \| (T - T_N)(x) \|^2_{\ell^2}$$

$$= \sup_{\|x\| \leq 1} \sum_{j \geq N} | \alpha_j x_j |^2$$

$$= \underbrace{\sup_{j \geq N} |\alpha_j|}_{0} \sup_{\|x\| \leq 1} \underbrace{\sum_{j \geq N} |x_j|^2}_{\leq \|x\|} \to 0$$

$\Rightarrow$ by Cor 2 T is compact

(ii) Integral operators on $C([0,1])$

Take $K \in C([0,1]^2)$, put

$$T : C([0,1]) \longrightarrow C([0,1])$$
$$f \longmapsto (Tf)(x) = \int_0^1 K(x,y) f(y)\,dy$$

Then T is compact

proof We show $T\, B_1^{C([0,1])}(0)$ is precompact. Apply A-A with

$$M = \{ g = Tf \; ; \; \|f\|_\infty \leq 1 \}$$

- M bounded

$\forall g \in M$

$$\|Tf\|_{L^\infty} \leq \sup_x \int_0^1 |K(x,y)|\, |f(y)|\, dy$$
$$\leq \sup_{x,y\in[0,1]} |K(x,y)| \, \|f\|_{L^\infty} \leq C$$

- M uniformly equicontinuous

Let $\varepsilon > 0$

$$|(Tf)(x_1) - (Tf)(x_2)| \leq \int_0^1 |K(x_1,y) - K(x_2,y)|\, |f(y)|\, dy$$

K is continuous on compact set

$\Downarrow$

K is uniformly continuous

$\Downarrow$

$\forall \varepsilon > 0 \; \exists \delta > 0 : \; |K(x_1,y) - K(x_2,y)| < \varepsilon \quad \forall\, |x_1 - x_2| < \delta$

$$\leq \varepsilon \, \|f\|_{L^\infty} \leq \varepsilon \quad \forall \|f\|_\infty \leq 1$$

Apply A.A. $\Rightarrow$ M is precompact $\Rightarrow$ T compact

(iii) <u>Integral Hilbert - Schmidt operators</u>

$K(x,y) \in L^2([0,1]^2)$, $T : L^2[0,1] \to L^2[0,1]$

$f \longmapsto (Tf)(x) = \int_0^1 K(x,y) f(y) dy$

T is compact op

<u>proof</u> Recall that $\|T\|_{\mathcal{L}(L^2)} \leq \|K\|_{L^2([0,1]^2)}$

We approximate T with finite rank operators

$$T_n f(x) = \int_0^1 K_n(x,y) f(y) dy$$

with $K_n(x,y) = \sum_{i,j \leq n} c_{ij} \varphi_i(x) \psi_j(y)$

$$(T_n f)(x) = \sum_{i,j \leq n} c_{ij} \varphi_i(x) \int_0^1 \psi_j(y) f(y) dy$$

$$= \sum_{i,j \leq n} c_{ij} \varphi_i(x) \langle f, \overline{\psi_j} \rangle$$

T_n is bd op $\|T_n\| \leq \|K_n\|_{L^2([0,1]^2)}$ ✓

(provided $\varphi_i, \psi_j \in L^2 \ \forall i,j$)

T_n is finite rank : $\mathrm{Im}\, T_n \subseteq \mathrm{span} \langle \varphi_1, \dots, \varphi_n \rangle$

How to choose φ_i, ψ_j?

$$\|T_n - T\|_{\mathcal{L}(L^2)} \leq \|K_n - K\|_{L^2([0,1]^2)}$$

$L^2[0,1]$ Hilb. space, it has an orthonormal basis $\varphi_j(t)$

$\Rightarrow$ $L^2([0,1]^2)$ has orth. basis $\varphi_j \otimes \varphi_\ell$ $\left(\varphi_j(x)\varphi_\ell(y) \right)_{j,\ell}$

Choose $\sum c_{ij}\, \varphi_i(x)\, \varphi_\ell(y)$ which approximate $K(x,y)$ in $L^2([0,1]^2)$

(iv) <u>Sobolev embedding</u> Ω bd set with smooth $\partial\Omega$

Then $H^1(\Omega) \hookrightarrow L^2(\Omega)$ in a compact way.

$i : H^1(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$

<u>Recall</u> $L^2 = \left\{ (\hat{v}_k)_{k \geq 1} \text{ Four. coeff.: } \sum_{k \in \mathbb{Z}} |\hat{v}_k|^2 < \infty \right\}$

$H^1 = \left\{ (\hat{v}_k)_{k \geq 1} \quad \text{"} \quad \text{"} \quad \sum_{k \in \mathbb{Z}} (1 + k^2) |\hat{v}_k|^2 < \infty \right\}$

<u>FACT</u>: $S \subseteq \ell^2$ compact $\iff$ (i) closed

(ii) bounded

(iii) $\lim_{N \to \infty} \sup_{x \in S} \sum_{i=N}^{+\infty} |x_i|^2 = 0$

Apply it with $S = \overline{i\, B_1^{H^1}(0)} = \left\{ (\hat{v}_k) : \sum (1+k^2) |\hat{v}_k|^2 \leq 1 \right\}$

(i) closed ✓

(ii) bd ✓ $\left(\|u\|_{L^2} \leq \|u\|_{H^1} \right)$

(iii) $$\sum_{i=N}^{+\infty} |\hat{v}_k|^2 = \sum_{k=N}^{+\infty} \frac{(1+k^2)}{1+k^2} |\hat{v}_k|^2$$

$$k \geq N \Rightarrow \frac{1}{1+k^2} \leq \frac{1}{1+N^2} \qquad \leq \frac{1}{1+N^2} \sum_{k \in \mathbb{Z}}^{+\infty} (1+k^2) |\hat{v}_k|^2$$

$$\leq \frac{1}{1+N^2} \|v\|_{H^1}$$

$\rightsquigarrow$ Take $\sup_{x \in S}$ ✓